10. Qubitization: Block Encodings

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Qubitization: Block-encoding framework

$$Z(\phi_0) \quad W(H) \quad Z(\phi_1) \quad W(H) \quad Z(\phi_2) \quad W(H) \quad Z(\phi_3)$$

Cost $\approx$ Cost of $W(H) \times$ Degree of the polynomial

Cost of $W(H) \approx 2 \times$ Cost of controlled-$U(H)$

Block Encoding Revisited

$$U(H)\,|\,G\rangle_a\,|\,\lambda\rangle_s = \lambda\,|\,G\rangle_a\,|\,\lambda\rangle_s + \sqrt{1-\lambda^2}\,|\,G_\lambda^\perp\rangle_{as},$$

where $\left(\langle G\,|_a \otimes I_s\right)|\,G_\lambda\rangle_{as}^\perp = 0.$

Main question: How do we actually implement $U(H)$?

Toy Model: Fermions

$$H = -t \sum_i (a_i^\dagger a_{i+1} + h.c.) + U \sum_i \hat{n}_i \hat{n}_{i+1} .$$

Toy Model: Spins

- After Jordan-Wigner transformation, we obtain:

$$H = -\frac{t}{2}\sum_i (X_i X_{i+1} + Y_i Y_{i+1}) + U\sum_i \frac{(Z_i + 1)(Z_{i+1} + 1)}{4}\,.$$

LCU

- Viewing H as a linear combination of unitaries, we see that there are $4N$ Pauli operators in total.

$$H = -\frac{t}{2}\sum_{i=1}^{N}(X_iX_{i+1} + Y_iY_{i+1}) + U\sum_{i=1}^{N}\frac{(Z_i + 1)(Z_{i+1} + 1)}{4}.$$

For concreteness, let's use the following convention:

$$P_k = \begin{cases} X_iX_{i+1} & : 0 \le k < N \\ Y_iY_{i+1} & : N \le k < 2N \\ Z_iZ_{i+1} & : 2N \le k < 3N \\ Z_i & : 3N \le k < 4N \end{cases}$$

(Set $i = k \mod N$.)

$$U(H)|G\rangle_a|x\rangle_s = x|G\rangle_a|x\rangle_s + \sqrt{1-x^2}\,|G_x^{\perp}\rangle_{as}$$

$$U(H) = \begin{pmatrix} H & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\|H\| \le \frac{|t|}{2}(\underbrace{N}_{X_iX_{i+1}} + \underbrace{N}_{Y_iY_{i+1}}) + \frac{|U|}{4}(2+1) = N\left(|t| + \frac{3}{4}|U|\right) = \alpha$$

$$\left\|\frac{H}{\alpha}\right\| \le 1$$

SELECT

- Viewing H as a linear combination of unitaries, we see that there are $4N$ Pauli operators in total.

$$H = -\frac{t}{2}\sum_i (X_iX_{i+1} + Y_iY_{i+1}) + U\sum_i \frac{(Z_i+1)(Z_{i+1}+1)}{4}.$$

For concreteness, let's use the following convention:

$$P_k = \begin{cases} X_iX_{i+1} & : 0 \le k < N \\ Y_iY_{i+1} & : N \le k < 2N \\ Z_iZ_{i+1} & : 2N \le k < 3N \\ Z_i & : 3N \le k < 4N \end{cases}$$

(Set $i = k \mod N$.)

$\mathrm{Sel}(H)\,|k\rangle_a|\psi\rangle_s = |k\rangle_a P_k|\psi\rangle_s$

Cost of $\mathrm{Sel}(H) = O$(# of terms in the Hamiltonian)

$= O(N)$

PREPARE

- Viewing H as a linear combination of unitaries, we see that there are $4N$ Pauli operators in total.

$$H = -\frac{t}{2}\sum_i (X_iX_{i+1} + Y_iY_{i+1}) + U\sum_i \frac{(Z_i+1)(Z_{i+1}+1)}{4}.$$

For concreteness, let's use the following convention:

$$P_k = \begin{cases} X_iX_{i+1} & : 0 \le k < N \\ Y_iY_{i+1} & : N \le k < 2N \\ Z_iZ_{i+1} & : 2N \le k < 3N \\ Z_i & : 3N \le k < 4N \end{cases}$$

(Set $i = k \mod N$.)

$$\mathrm{Prep}(H)\,|0\cdots0\rangle_a = \sum_{i=0}^{4N-1} \frac{\sqrt{\alpha_i}\,|i\rangle_a}{\sqrt{\|\vec{\alpha}\|_1}}$$

$$\vec{\alpha} = (\alpha_0, \cdots \alpha_{4N-1})$$

$$\alpha_k = \begin{cases} -\frac{t}{2} & \text{for } 0 \le k < 2N \\ \frac{U}{4} & \text{for } 2N \le k < 3N \\ \frac{1}{2}U & \text{for } 3N \le k < 4N \end{cases}$$

SELECT+PREPARE

- Viewing H as a linear combination of unitaries, we see that there are $4N$ Pauli operators in total.

$$H = -\frac{t}{2}\sum_i (X_iX_{i+1} + Y_iY_{i+1}) + U\sum_i \frac{(Z_i+1)(Z_{i+1}+1)}{4}.$$

For concreteness, let's use the following convention:

$$P_k = \begin{cases} X_iX_{i+1} & : 0 \le k < N \\ Y_iY_{i+1} & : N \le k < 2N \\ Z_iZ_{i+1} & : 2N \le k < 3N \\ Z_i & : 3N \le k < 4N \end{cases}$$

(Set $i = k \mod N$.)

$$\langle 0|_a \mathrm{Prep}(H)^\dagger \, \mathrm{Sel}(H) \, \mathrm{Prep}(H) \, |0\rangle_a |\psi\rangle_s$$

$$\mathrm{Prep}(H)|0\rangle_a = \sum_k \sqrt{\alpha_k}\,|k\rangle \Big/ \sqrt{\|\vec{\alpha}\|_1}$$

$$\sum_k \mathrm{Sel}(H) \frac{\sqrt{\alpha_k}}{\sqrt{\|\vec{\alpha}\|_1}} |k\rangle |\psi\rangle_s = \sum_k \frac{\sqrt{\alpha_k}}{\sqrt{\|\vec{\alpha}\|_1}} |k\rangle P_k |\psi\rangle_s$$

$$\langle 0|_a \mathrm{Prep}(H)^\dagger \mathrm{Sel}(H) \mathrm{Prep}(H) \, |0\rangle_a |\psi\rangle_s = \sum_k \frac{|\alpha_k|}{\sqrt{\|\vec{\alpha}\|_1}} P_k |\psi\rangle_s$$

$$\mathrm{Prep}(H)^\dagger \mathrm{Sel}(H) \mathrm{Prep}(H) \, |0\rangle_a |\psi\rangle_s = |0\rangle_a \sum_k \frac{|\alpha_k|}{\cdots} P_k |\psi\rangle_s + |G_\psi^\perp\rangle$$

Sign

- Viewing H as a linear combination of unitaries, we see that there are $4N$ Pauli operators in total.

$$H = -\frac{t}{2}\sum_i (X_i X_{i+1} + Y_i Y_{i+1}) + U \sum_i \frac{(Z_i + 1)(Z_{i+1} + 1)}{4}.$$

For concreteness, let's use the following convention:

$$P_k = \begin{cases} (-1)^a X_i X_{i+1} & : 0 \le k < N \\ (-1)^a Y_i Y_{i+1} & : N \le k < 2N \\ (-1)^b Z_i Z_{i+1} & : 2N \le k < 3N \\ (-1)^b Z_i & : 3N \le k < 4N \end{cases}$$

$\langle 0|_a \mathrm{Prep}(H)^\dagger \mathrm{Sel}(H)\, \mathrm{Prep}(H)\, |0\rangle_a |\psi\rangle_s = \sum_k \frac{|\alpha_k|}{\sqrt{\|\vec{\alpha}\|_1}} P_k |\psi\rangle_s$

$H \propto \sum_k \alpha_k P_k$

(Set $i = k \mod N$.)

$|0\rangle|\psi\rangle \xrightarrow{U} |0\rangle|\psi\rangle$

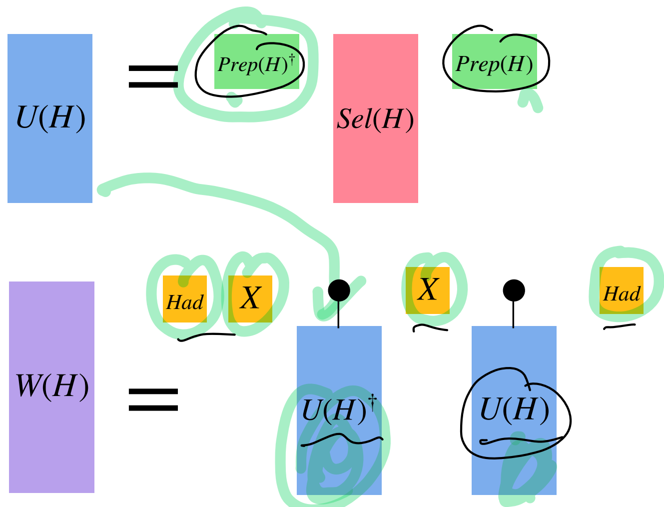
$|1\rangle|\psi\rangle \xrightarrow{U} |1\rangle P_k|\psi\rangle$

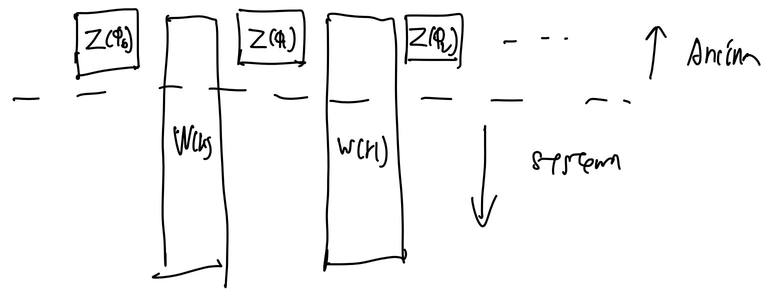
Change U to $Z_q U$

$|0\rangle|\psi\rangle \xrightarrow{Z_q U} |0\rangle|\psi\rangle$

$|1\rangle|\psi\rangle \xrightarrow{Z_q U} -|1\rangle P_k|\psi\rangle$

Qubitization: The gate sequence





Cost $\approx (4 \times$ controlled-$Prep(H) + 2 \times$ controlled-$Sel(H)) \times$ Degree of the polynomial

$W(H)|G\rangle_a|\lambda\rangle_s = \lambda|G\rangle_a|\lambda\rangle_s + \sqrt{1-\lambda^2}|G^\perp\rangle_{as}$

Other Examples

Quantum Singular Value Decomposition $A = USV^\dagger$

While the SELECT+PREPARE framework is very useful in the context of quantum simulation, there are other examples. [Low and Chuang (2017)]

$Ax=b \quad A' = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \quad A'x = b'$

1. Sparse Matrix: Solving systems of linear equation [Harrow, Hassidim, and Lloyd (2009)]

2. Density matrix encoding : Useful for quantum principal component analysis [Lloyd, Mohseni, and Robentrost (2013)]

Let's talk about density matrix encoding, for concreteness.

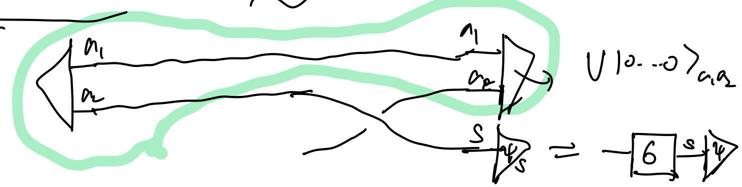
$|\psi\rangle \to e^{i\sigma t}|\psi\rangle$ $\quad\sigma$: Anything, as long as it is easy to prepare its purification

$U|0\cdots0\rangle_{a_1a_2}$

$a = a_1a_2$ s.t. $\mathrm{Tr}_{a_2}(U|0\cdots0\rangle_{a_1a_2}\langle 0\cdots0|_{a_1a_2}U^\dagger) = \sigma_{a_1}$

$\mathrm{SWAP}_{a_1 s}U|0\cdots0\rangle_{a_1a_2}|\psi\rangle_s = U|0\cdots0\rangle_{a_1a_2}\,\sigma|\psi\rangle_s + |G^\perp\rangle$

$\langle 0\cdots 0|_{a_1a_2} U^\dagger \mathrm{SWAP}_{a_2,s} U |0\cdots 0\rangle_{a_1a_2} |\psi\rangle_s$



Quantum Phase Estimation (QPE) revisited

$|\psi\rangle = \sum_\lambda \alpha_\lambda |\psi_\lambda\rangle$ $\xrightarrow{\text{QPE}}$ Get eigenstate $|\psi_\lambda\rangle$ w. probability $= |\alpha_\lambda|^2$

→ eigenstates of H

Recall that QPE leverages an ability to synthesize time evolution e^{-iHt} to perform (i) eigenvalue estimation and (ii) eigenstate preparation.

Note that H and $\arccos(H)$ have the same eigenstates and their eigenvalues are related by $\lambda \leftrightarrow \cos^{-1}(\lambda)$ (assuming eigenvalues ≤ 1). Moreover, $e^{i\cos^{-1}(H)}$ can be implemented *exactly* using qubitization. It turns out that the QPE using $e^{i\cos^{-1}(H)}$ is more efficient.

$\|H\| \leq 1$

$e^{i\cos^{-1}H} = \cos(\cos^{-1}H) + i\sin(\cos^{-1}H) = H + \sqrt{I - H^2}$

So for QPE, we don't even need to implement time evolution!

[Poulin et al. (2017), Berry et al. (2017)]

Summary

Qubitization is a very flexible modern framework for developing quantum algorithms.

While the unitary encoding we discussed last time is somewhat simplistic, it captures the essential ideas.

Many of the recent advances in Hamiltonian simulation algorithms use the framework of qubitization. Improvements were made in SELECT+PREPARE subroutine, which utilizes the special structure of the Hamiltonian.